NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Wednesday May 11, 1983

Time allowed - 3½ hours

Write your name on each page. Start each question on a new sheet.

Important Notice: In the selection of the IMO team special attention will be paid to performance in geometry. Two at least of the questions answered should be from G1, G2, G3. The marks from not more than five questions will count towards your score.

G1. Two points A, B and a line k are given in a plane. Locate, with proof, the point P of the plane for which $PA^2 + PB^2 + PN^2$ is a minimum, where N is the foot of the perpendicular from P to k.

Give a generalisation without proof for three points A, B, C and $PA^2 + PB^2 + PC^2 + PN^2$ a minimum.

G2. Consider the three escribed circles of the triangle ABC, that is, the three distinct circles each of which touches one side of triangle ABC internally and the other two externally. Each pair of escribed circles has just one common tangent which is not a side of triangle ABC, and the three such common tangents form a triangle T.

O is the circumcentre of triangle ABC. Prove that OA is perpendicular to a side of T.

G3. ℓ, m, n are three lines in space. Neither ℓ nor m is perpendicular to n. Points P and Q vary on ℓ and m respectively in such a way that PQ is perpendicular to n. The plane through P perpendicular to m meets n at R and the plane through Q perpendicular to ℓ meets n at S. Prove that RS is of constant length.

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A4. Prove that if a, b, c, d, e, f are positive real numbers then

$$\frac{ab}{a+b} + \frac{cd}{c+d} + \frac{ef}{e+f} \leqslant \frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

A5. Find the number of arrangements

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h$$

of the numbers 1,2,3,4,5,6,7,8 which satisfy all seven conditions

$a<b$, $c<d$, $e<f$, $g<h$ and $b>c$, $d>e$, $f>g$.

A6. n and k are positive integers. Find all pairs (n,k) satisfying

$$(n+1)^k = n! + 1,$$

proving that you have the complete set of solutions.

A7. In a colony of $(mn+1)$ mice, prove that at least one of the following statements is true:

(a) There is a set A of $(m+1)$ mice none of which is a parent of any other in the set.

(b) There is an ordered set B of $(n+1)$ mice $a_1, a_2, \ldots, a_n, a_{n+1}$ such that a_{i+1} is a parent of a_i for each $i = 1,2,\ldots,n$.